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Structured Latent Curve Models for the Study of Change in Multivariate Repeated Measures

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This article considers a structured latent curve model for multiple repeated measures. In a structured latent curve model, a smooth nonlinear function characterizes the mean response. A first-order Taylor polynomial taken with regard to the mean function defines elements of a restricted factor matrix that may include parameters that enter nonlinearly. Similar to factor scores, random coefficients are combined with the factor matrix to produce individual latent curves that need not follow the same form as the mean curve. Here the associations between change characteristics in multiple repeated measures are studied. A factor analysis model for covariates is included as a means of relating latent covariates to the factors characterizing change in different repeated measures. An example is provided.

In studies of repeated measures, investigators often obtain measures of two or more variables with interest in how each variable changes across the study period. Although most studies of change have focused on analysis of a single response variable, recent applications of popular analytic methods have considered the joint observation of multiple response variables measured at multiple time points. In these applications, researchers may be interested not only in the assessment of individual differences in change in individual variables but also in how characteristics of change in different variables are related. This may occur in a variety of situations. In one setting, the same group of individuals provides measures of the variables of interest, such as in studies of physical or psychological well-being wherein multiple indicators of health are obtained. It is interesting then to determine whether, for example, an increase in one aspect of health is related to an increase (or decrease) in another.

In a slightly different setting, multiple informants may provide information about a key individual, such as parents and teachers providing developmental information about a child, and it is of interest to study the degree of agreement (or disagreement) between the different sources assessed across time. Another situation involves the study of distinguishable individuals nested within pairs (e.g., men and women nested within couples) or individuals nested within slightly larger groups (e.g., individuals nested within households), in which interest may be in the extent to which individuals within groups have similar ratings on the variables of interest. Many other situations are, of course, possible. For any such case in a cross-sectional setting, there is often interest in how the variables relate to each other at the one point in time. With the addition of multiple time points, it is then interesting to also consider how characteristics of change in the different variables may be related.

Mixed-effects models, also known as multilevel, random coefficient, or hierarchical linear models, have been used extensively to describe data in which the units of study are nested within higher level units (Bock, 1989; Bryk & Raudenbush, 1987; Goldstein, 1995; Kreft & de Leeuw, 1998; Longford, 1993; Snijders & Bosker, 1999). A classic example comes from education, in which students are the unit of study and are nested within classrooms or schools. These models have been successfully applied to repeated measures data in which repeated observations of the same variable are recorded over a specified period of time or space for a sample of individuals. For time structured data, a linear mixed-effects model is composed of a fixed part that characterizes the response at the population level, providing a summary of how a variable changes on average as a function of time. The random part is handled as a two-stage procedure, yielding information about variation in re-

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A modified version of the data file appears on my Web site: http://psychology.ucdavis.edu/labs/Blozis/index.htm. Additional materials [SAS code for implementing a version of the model] are on the Web at http://dx.doi.org/10.1037/1082-989X.9.3.334.supp.

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sponses at the level of the individual (Level 1) as well as variation in response trajectories between individuals (Level 2).

Closely related to mixed-effects models are latent growth models, or latent curve models. The latent curve model (Meredith & Tisak, 1984, 1990) is based on seminal work produced independently by Rao (1958), Tucker (1958, 1966), and Scher, Young, and Meredith (1960, appendix). In this earlier work, a technique similar to component analysis was used to produce observable weights akin to component scores. The procedure has been applied to learning and growth data. In these settings, components that characterize the shape of the curve are combined with the weights to obtain individual growth or learning curves. As a descriptive procedure, however, the result is not a testable model. In contrast, the latent curve model is similar to the standard unrestricted factor analysis model in which both the mean vector and the covariance matrix of the manifest variables have imposed structures. In a latent curve model, columns of the factor matrix define the general shape of the response trajectories over time and may be specified in advance or left to be estimated, with the latter approach providing some degree of flexibility in describing the mean curve when a structure cannot be specified in advance or adequately summarized by conventional structures such as a polynomial function (McArdle, 1988). This is in contrast to a linear mixed-effects model that is usually based on a prespecified mean structure, such as a straight line or quadratic function. The factors, or random coefficients, define the characteristics of change in the response. Individual latent response curves are assumed to be a linear combination of the factor matrix and the random coefficients specific to the individual. The random coefficients permit the individual latent curves to deviate from the mean curve, allowing for individual differences in response trajectories.

Individual differences in characteristics of change present the possibility of their study by introducing predictors or correlates of the random coefficients at the second level of the model. Latent covariates can be handled by including a factor analysis model for a subset of variables wherein the common factor is related to the random coefficients at the second level (Browne, 1993). Blozis and Cudeck (1999) developed a partially nonlinear mixed-effects model, a form of mixed-effects model in which fixed parameters may enter nonlinearly but random coefficients are restricted to enter linearly, with a factor analysis model for covariates measured with error. Latent covariates are then studied as correlates of the random coefficients in the second level of the model.

An appealing characteristic of the random coefficient model in a repeated measures setting is that it may be used to analyze data that are incomplete or for which individuals are observed at different time points (Jennrich & Schluchter, 1986; McArdle & Hamagami, 1992; Mehta & West, 2000). Latent curve models, based on a factor analysis model, originally depended on estimation techniques involving the use of sufficient statistics, such as a sample mean vector and covariance matrix. Estimation procedures that make use of raw data allow for these models to be fitted to incomplete or unbalanced data (e.g., Jennrich & Schluchter, 1986). This is particularly useful in analyses of longitudinal data, in which data are often incomplete or individuals are observed at different time points.

More recently, these models have been considered for the analysis of multivariate repeated measures (MacCallum, Kim, Malarkey, & Kiecolt-Glaser, 1997). Reinsel (1992, 1994) considered a linear mixed-effects model for multivariate normal repeated measures that are both complete and balanced with respect to measurement occasions. Shah, Laird, and Schoenfeld (1997) considered a linear mixed-effects model for multivariate repeated measures that are incomplete, unbalanced, or both. Estimation of these models is generally straightforward, relying on techniques common to the univariate case. To date, several applications of random coefficient models for multivariate repeated measures have appeared (e.g., Curran, Stice, & Chassin, 1997; McArdle & Anderson, 1990). A nonlinear form of the model in which random coefficients enter the model nonlinearly has also been considered for multivariate repeated measures (Davidian & Giltinan, 1995). Although more flexible than the linear version, the fully nonlinear mixed-effects models can be difficult to estimate; however, efforts in this area have produced promising results (Cudeck & du Toit, 2003; Davidian & Giltinan, 1995; Pinheiro & Bates, 2000). The difficulty arises when the random coefficients enter the model in a nonlinear way, resulting in a marginal distribution of the response that cannot, in most cases, be evaluated (Davidian & Giltinan, 1995, Section 4.4).

This article considers a structured latent curve model for multivariate normal repeated measures that also includes a factor analysis model for a separate set of variables that are correlates of the random coefficients in the second level of the latent curve model. This article does not propose a new technique but rather exposes researchers to the application of structured latent curve models in a multivariate setting (MacCallum et al., 1997). The focus is on a model that allows for the study of the associations between characteristics of change in different response variables whose mean curves may be nonlinear in form, as well as the associations between these characteristics and a separate set of variables (possibly latent) considered to be related to the change processes of the repeated measures at the second level. Estimation of the model proceeds in a straightforward manner using techniques standard for linear models. This is due to the fact that the parameters defining the factor matrix may enter in a nonlinear manner but are assumed to be fixed across individuals. The random coefficients, which vary across individuals, enter the model linearly. When the random coefficients enter the model in a strictly linear manner, it is possible to estimate a nonlinear function by normal maximum likelihood with methods typically used for linear models (Cudeck & du Toit, 2003; Davidian & Giltinan, 1995). Thus, a benefit of the structured latent curve model is that many nonlinear forms of change are easily handled.

A common approach to approximating a nonlinear response form is to use a polynomial. There have been several successful applications in which a quadratic growth model was used (see, e.g., Bryk & Raudenbush, 1992; Raudenbush, Brennan, & Barnett, 1995; Windle & Windle, 2001). Models based on polynomial functions are generally easy to estimate because the random coefficients enter in a linear manner. Sometimes it is necessary to expand the model by including terms that represent different powers of the predictors. This may be done to better approximate nonlinear response forms. In a model for the study of change, these additional terms typically involve transformations of the time variable.

There are, however, some drawbacks to this approach that are not necessarily applicable to the methods discussed here. With an increase in the number of power terms, the curve becomes a better approximation to the observed responses but may also begin to serve as a model for random variation rather than the true shape of the response (Weisberg, 1985). As discussed by Pinheiro and Bates (2000), with an increase in the number of terms there is also a decrease in parsimony and, in general, the possibility of multicollinearity among coefficients. In many cases, a polynomial, such as a quadratic, is not suitable as a means of describing the behavior under study. For example, in learning studies performance measures may show signs of stability after the participant has mastered the task. In a treatment setting, there may be a lag in response to the treatment, with the treatment effect occurring over the latter phase of the study period. In the case of these response patterns, a parabolic form is not ideal. Responses of these forms could possibly be handled by spline regression models in which different segments of the response period are modeled with different polynomial functions (Cudeck & Klebe, 2002; Snijders & Bosker, 1999). Another strategy is to use a nonlinear function to characterize change. Nonlinear functions may be specified to have parameters directly relevant to the behavior, making the model highly interpretable and parsimonious (Cudeck, 1996; Pinheiro & Bates, 2000). This is especially important in subsequent studies of individual differences in change characteristics.

Before development of the structured latent curve model for multiple repeated measures, descriptions of the latent curve and the structured latent curve models for a single repeated measure are provided. Building on Browne (1993) and extending the work of MacCallum et al. (1997), the structured latent curve model for multiple repeated measures that also includes a factor analysis model for covariates is then described, followed by an application to data from a learning experiment.

Background: Latent Curve Models for a Single Repeated Measure

The latent curve model of Meredith and Tisak (1984, 1990) has received extensive consideration (MacCallum et al., 1997; McArdle & Epstein, 1987; Stoolmiller, 1995; Willett & Sayer, 1994). The model is reviewed here. Considering a set of repeated observations of a random variable Y for individual i, let the vector $\mathbf{y}_i = (y_{1i}, y_{2i}, \ldots, y_{Ti})'$ be a set of responses on Y for individual i. The responses are observed according to a set of measurement occasions $\mathbf{t}_i = (1, \ldots, T_i)'$, where T_i is the total number of observations for the individual. As suggested by the subscript i on $\mathbf{t}_i$, times of measurement may vary from one person to another as a result of, for example, individuals being assessed at different measurement occasions or on a different number of occasions. In a latent curve model, it is typically assumed that true change in the response variable follows a specified form common to all individuals, such as a straight line, but that the individual curves may vary from one another, such as in their intercepts and linear rates of change. The errors of measurement corresponding to each measurement occasion are often assumed to be mutually independent and distributed with constant variance across time.

A general expression for the latent curve model for $\mathbf{y}_i$ is

$$\mathbf{y}_i = \boldsymbol{\Lambda}_i \boldsymbol{\eta}_i + \boldsymbol{\epsilon}_i, \tag{1}$$

where $\boldsymbol{\Lambda}_i$ is a factor loading matrix:

$$\boldsymbol{\Lambda}_i = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1J} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2J} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{Ti1} & \lambda_{Ti2} & \cdots & \lambda_{TiJ} \end{bmatrix}.$$

The number of rows in $\boldsymbol{\Lambda}_i$ is equal to the number of measurement occasions on which individual i was observed. The columns of $\boldsymbol{\Lambda}_i$, commonly referred to as "basis" curves, have elements (e.g., λ_{11}) that define the shape of the curve over the observed measurement occasions. For example, in fitting a model in which the response is a linear function of time, the factor matrix typically contains a column of ones for the intercept and a column of fixed values equal to the measures of time (Willett & Sayer, 1994). To fit a nonlinear curve, nonlinear transformations of time may be introduced, such as a quadratic or cubic transformation. Otherwise, in some cases certain elements of the factor loading matrix may be estimated to handle other forms of nonlinear change (McArdle, 1988; Meredith & Tisak, 1990; Stoolmiller, 1995).

The factor $\boldsymbol{\eta}_i = (\eta_{1i}, \eta_{2i}, \ldots, \eta_{Ji})'$ is a vector of random coefficients particular to individual i. The random coefficients represent different characteristics of change. Combined with the factor matrix in Equation 1, an individual's set of random coefficients indicates the extent to which that person's latent response curve depends on the basis curves. Using the example of a model for which the response is a linear function of time, a relatively large value for the random intercept, would, for instance, indicate an intercept that is high relative to others. The mean and covariance matrix of the random coefficients are $\boldsymbol{\alpha}$ and $\boldsymbol{\Psi}$, respectively. The means of the random coefficients represent the characteristics of change in the average response, that is, change at the population level. The covariance matrix $\boldsymbol{\Psi}$ is

$$\boldsymbol{\Psi} = \begin{bmatrix} \sigma^2_{\eta_1} & & & \\ \sigma_{\eta_2\eta_1} & \sigma^2_{\eta_2} & & \\ \vdots & \vdots & \ddots & \\ \sigma_{\eta_J\eta_1} & \sigma_{\eta_J\eta_2} & \cdots & \sigma^2_{\eta_J} \end{bmatrix},$$

where the diagonal elements are the variances of the random coefficients and the off-diagonal elements are the covariances between them. The variances of the coefficients are measures of the extent to which individuals differ in each change characteristic, and the covariances between them represent the linear associations between different characteristics. For an individual, the combination of the factor loading matrix and the random coefficients yields a latent curve. That is, true change in an individual's response is a function of a factor matrix common to all individuals but is weighted differently by change characteristics that may vary from one person to another. Finally, the measurement errors $\boldsymbol{\epsilon}_i = (\epsilon_{1i}, \epsilon_{2i}, \ldots, \epsilon_{Ti})'$ are discrepancies between the person's true curve and his or her observed curve at different time points. The errors are often assumed to have means equal to zero and covariance matrix $\boldsymbol{\Theta}_{\epsilon i}$, where $\boldsymbol{\Theta}_{\epsilon i}$ depends on the individual only with regard to its dimensions. Typically, the errors are assumed to be independent between measurement occasions with constant variance across time, taking on the following structure:

$$\boldsymbol{\Theta}_{\epsilon i} = \begin{bmatrix} \sigma^2_{\epsilon} & & & \\ & \sigma^2_{\epsilon} & & \\ & & \ddots & \\ & & & \sigma^2_{\epsilon} \end{bmatrix},$$

where σ^2_{ϵ} is equal to the variance of the measurement errors: $var(\epsilon_i)$. Finally, the measurement errors and the random coefficients are assumed to be independent. Given the preceding assumptions, the mean vector and covariance matrix of $\mathbf{y}_i$ are

$$\boldsymbol{\mu}_i = \boldsymbol{\Lambda}_i \boldsymbol{\alpha}$$

and

$$\boldsymbol{\Sigma}_i = \boldsymbol{\Lambda}_i \boldsymbol{\Psi} \boldsymbol{\Lambda}_i' + \boldsymbol{\Theta}_{\epsilon i}.$$

In most cases, the covariance matrix of the random coefficients is assumed to be unstructured so that the variances of the different coefficients may differ and the coefficients may covary. At the individual level, the error structure $\boldsymbol{\Theta}_{\epsilon i}$ may assume a variety of forms other than one that assumes independence between time points and constant variance across time. Willett and Sayer (1994) provided several examples for different within-individual error structures, such as one that assumes independence but heterogeneity of variance across time.

In a latent curve model, the form of the curve need not be explicitly defined. That is, it is not necessary to specify in advance all elements of the factor matrix that define the different aspects of change in a response variable (McArdle, 1988; Meredith & Tisak, 1990; Stoolmiller, 1995). The advantage is that the model can handle some nonlinear forms of change, making it a useful method for certain types of data, such as developmental, growth, or learning data, that typically do not follow a linear trajectory. On the basis of a latent curve model, a structured latent curve model specifies in advance elements of the factor matrix; parameters of the matrix may enter nonlinearly with the assumption that they are fixed across individuals. Similar to the case with the standard latent curve model, the random coefficients of the structured latent curve model enter in a linear manner, thus allowing for estimation of the model with common techniques such as normal maximum likelihood. A description of the structured latent curve model for a single repeated measure is provided here. Readers are referred to Browne (1993) for more details concerning the development of the model.

Structured Latent Curve Models for a Single Repeated Measure

Formulation of the structured latent curve model begins with a specification of a function, referred to as the target function, to represent the mean response. The curves of individuals, however, need not follow the form of the target function. In discussing the more technical aspects of the model, it is convenient to first consider the case in which data are complete and all individuals are observed according to the same data collection scheme. Cases for which data are unbalanced or missing are addressed later, when the model is considered for multiple repeated measures.

First, let $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_T)'$ denote the set of true mean responses for a random variable Y observed according to a set of discrete measurement occasions, $\mathbf{t} = (1, 2, \ldots, T)'$. In a structured latent curve model, the mean curve is assumed to follow a smooth function, $f(\boldsymbol{\theta}, \mathbf{t})$, where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_J)'$ is a vector of J unknown, fixed parameters. That is,

$$\begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_T \end{bmatrix} = \begin{bmatrix} f_1(\theta, 1) \\ f_2(\theta, 2) \\ \vdots \\ f_T(\theta, T) \end{bmatrix},$$

where $f_t(\theta, t)$ represents the target function evaluated at time t. For example, Meredith and Tisak (1990, p. 117) used an exponential function to characterize mean response times on a learning task:

$$\mu_t = f(\theta, t) = \theta_1 - (\theta_1 - \theta_2)exp[-\theta_3(t-1)], \quad (2)$$

where, on occasion t, the mean response μ_t is assumed to be a function of three parameters: θ_1 and θ_2, representing population potential and initial response times, respectively, and θ_3, representing the population initial rate of change. The coefficients θ_1 and θ_2 enter the equation in a linear manner. The coefficient θ_3 enters nonlinearly.

The set of responses for an individual, $\mathbf{y}_i = (y_{1i}, \ldots, y_{Ti})'$, is assumed to be the sum of a common score vector $\mathbf{z}_i$ and an error vector $\boldsymbol{\epsilon}_i$:

$$\begin{bmatrix} y_{1i} \\ y_{2i} \\ \vdots \\ y_{Ti} \end{bmatrix} = \begin{bmatrix} z_{1i} \\ z_{2i} \\ \vdots \\ z_{Ti} \end{bmatrix} + \begin{bmatrix} \epsilon_{1i} \\ \epsilon_{2i} \\ \vdots \\ \epsilon_{Ti} \end{bmatrix}$$

$$\mathbf{y}_i = \mathbf{z}_i + \epsilon_i. \quad (3)$$

It is assumed that the expected value of the common score $\mathbf{z}_i$ is $\boldsymbol{\mu}$. It follows then, under the model for the mean response, that the expected value of the common score is equal to the function $\mathbf{f}(\boldsymbol{\theta}, \mathbf{t})$. That is,

$$E(\mathbf{z}_i) = \boldsymbol{\mu} = \mathbf{f}(\boldsymbol{\theta}, \mathbf{t}). \quad (4)$$

Thus, the form of the mean response specified in Equation 2, for example, concerns the form of change in the mean of the common scores and not the observed scores. It is also assumed that the common scores and the measurement errors are independent, such that $cov(\mathbf{z}_i, \boldsymbol{\epsilon}_i') = \mathbf{0}$. The error variate $\boldsymbol{\epsilon}_i$ is assumed to have an expected value of $\mathbf{0}$ and covariance matrix $\boldsymbol{\Theta}_{\epsilon i}$.

In the latent curve model of Equation 1, an individual's latent response is specified to be a linear, weighted combination of the basis curves. Following this setup for a structured latent curve model, a person's latent response is expressed as a linear, weighted combination based on a first-order Taylor polynomial taken with respect to the target function:

$$\mathbf{z}_i = \mathbf{f}(\boldsymbol{\theta}, \mathbf{t}) + \eta_1\mathbf{f}_1'(\boldsymbol{\theta}, \mathbf{t}) + \eta_2\mathbf{f}_2'(\boldsymbol{\theta}, \mathbf{t}) + \cdots + \eta_J\mathbf{f}_J'(\boldsymbol{\theta}, \mathbf{t}), \quad (5)$$

where $\mathbf{f}_j'(\boldsymbol{\theta}, \mathbf{t})$ is the first partial derivative of the target function $\mathbf{f}(\boldsymbol{\theta}, \mathbf{t})$ with respect to the jth parameter in $\boldsymbol{\theta}$:

$$\mathbf{f}_j'(\boldsymbol{\theta}, \mathbf{t}) = \frac{\partial \mathbf{f}(\theta, \mathbf{t})}{\partial \theta_j}.$$

The coefficient η_j is an individual-level deviate with expected value equal to zero, for $j = 1, 2, \ldots, J$. By setting the means of the individual coefficients η_j equal to zero, the expected value of the individual response curve in Equation 5 satisfies the expression in Equation 4; that is, $E(\mathbf{z}_i) = \mathbf{f}(\boldsymbol{\theta}, \mathbf{t})$. The partial derivatives of the target function given in Equation 5 then serve to make up the columns of a factor matrix $\boldsymbol{\Lambda}$:

$$\boldsymbol{\Lambda} = \begin{bmatrix} f_1'(\theta, 1) & f_2'(\theta, 1) & \cdots & f_J'(\theta, 1) \\ f_1'(\theta, 2) & f_2'(\theta, 2) & \cdots & f_J'(\theta, 2) \\ \vdots & \vdots & \ddots & \vdots \\ f_1'(\theta, T) & f_2'(\theta, T) & \cdots & f_J'(\theta, T) \end{bmatrix}. \quad (6)$$

Assuming that the common scores follow the expression of Equation 5 and the basis curves form the columns of $\boldsymbol{\Lambda}$ as in Equation 6, the observed response $\mathbf{y}_i$ in Equation 3 may be reexpressed as

$$\mathbf{y}_i = \mathbf{f}(\boldsymbol{\theta}, \mathbf{t}) + \boldsymbol{\Lambda\eta} + \boldsymbol{\epsilon}_i, \quad (7)$$

where $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_J)'$. The expression of Equation 7 does not, however, follow a latent curve model as defined in Equation 1 as a result of the inclusion of the target function, $\mathbf{f}(\boldsymbol{\theta}, \mathbf{t})$. Under the condition that the target function is invariant to a constant scaling factor, the target function can be decomposed into a matrix of its basis curves and the factor mean vector, $\boldsymbol{\alpha}$ (see Browne, 1993, p. 177), such as

$$\mathbf{f}(\boldsymbol{\theta}, \mathbf{t}) = \boldsymbol{\Lambda}\alpha.$$

Then the expression for $\mathbf{y}_i$ in Equation 7 may be reexpressed as

$$\mathbf{y}_i = \boldsymbol{\Lambda}\alpha + \boldsymbol{\Lambda\eta} + \boldsymbol{\epsilon}_i. \quad (8)$$

Next, assuming $\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \boldsymbol{\eta}$, the model in Equation 8 simplifies to

$$\mathbf{y}_i = \boldsymbol{\Lambda\eta}_i + \boldsymbol{\epsilon}_i,$$

where it is assumed that $E(\boldsymbol{\eta}_i) = \boldsymbol{\alpha}$. The factor mean vector $\boldsymbol{\alpha}$ may be obtained by solving the set of linear equations: $\mathbf{f}(\boldsymbol{\theta}, \mathbf{t}) = \boldsymbol{\Lambda\alpha}$. For the exponential function in Equation 2, $\boldsymbol{\alpha} = (\theta_1, \theta_2, 0)'$.

In summary, the population curve is assumed to follow the form of the target function $\mathbf{f}(\boldsymbol{\theta}, \mathbf{t})$. At the level of the individual, the random coefficient η_{ji} represents the dependence of the individual's true curve on the jth basis curve. The random coefficients in the vector $\boldsymbol{\eta}_i$, then, have the same interpretation as the fixed parameters in $\boldsymbol{\theta}$ as specific characteristics of change in the response. In addition, individual latent curves close to the mean curve will be similar in shape to the mean curve; those far from the mean curve can differ appreciably in shape.

Specification of a structured latent curve model is initially based on identifying a function suitable for summarizing the mean response, which is also invariant to a constant scaling factor. Browne (1993) provided details for fitting a structured latent curve model using three different functions: the Exponential, Logistic, and Gompertz. If all individuals are observed according to the same data collection scheme, then a plot of the sample means may be useful in identifying a suitable mean function. Choosing a function for the means may be more challenging when data are not balanced with respect to time, because the sample mean curve cannot be generated from the data. A plot of the individual curves may, in some cases, be helpful in identifying a reasonable mean structure; however, the shapes of the individual curves do not necessarily characterize the shape of the mean curve (Estes, 1956). Past experience with the behavioral response or other theoretical considerations might also suggest a reasonable mean form.

Structured Latent Curve Models for Multiple Repeated Measures

The latent curve model for a single response variable is extended for the simultaneous consideration of multiple response variables. The model presented in this section allows for different data collection schemes for different individuals, as was done in the earlier discussion of the latent curve model. First, let K be the number of variables for which repeated measures are taken. The variables may represent measures on the same individual, in which case the variables are nested within individuals, or different variables may be observed for distinguishable individuals related by group membership, such as men and women nested within couples or two parents and a target child nested within households. Here, let i denote this grouping unit, whether it be an individual or a group, and hereafter i is referred to as an individual for convenience.

For ease of presentation, it is assumed here that each individual has at least one observation for each repeated measures variable, although the method basically requires at minimum that an individual have data for at least one measure. Let $\mathbf{y}_{ki} = (y_{1ki}, \ldots, y_{Tki})'$ be individual i's set of responses on the kth response variable observed according to time points $\mathbf{t}_{ki} = (1, \ldots, T_{ki})$, where T_{ki} is the total number of observations on variable k for individual i, $k = 1, \ldots, K$. As in the univariate case, the measures of time for which any variable is observed may vary across individuals so that individuals are not necessarily measured at the same times. Further, the different response variables within individuals may be considered at different time points or have different missing data patterns. That is, there is no requirement that any of the variables, either within or between individuals, be measured according to the same time points. A multivariate response vector $\mathbf{y}_i$ is formed by stacking the response vectors corresponding to the individual measures:

$$\mathbf{y}_i = \begin{bmatrix} \mathbf{y}_{1i} \\ \mathbf{y}_{2i} \\ \vdots \\ \mathbf{y}_{Ki} \end{bmatrix}.$$

The set of responses in $\mathbf{y}_i$ is observed according to $\mathbf{t}_i$, where $\mathbf{t}_i$ is also formed by stacking the sets of time points specific to the individual:

$$\mathbf{t}_i = \begin{bmatrix} \mathbf{t}_{1i} \\ \mathbf{t}_{2i} \\ \vdots \\ \mathbf{t}_{Ki} \end{bmatrix}.$$

The total number of observations for individual i is the sum of all observations on all variables:

$$T_i = \sum_{k=1}^{K} T_{ki}.$$

The mean of this multivariate response vector is

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \\ \vdots \\ \boldsymbol{\mu}_K \end{bmatrix},$$

where $\boldsymbol{\mu}_k = (\mu_{1k}, \mu_{2k}, \ldots, \mu_{Tk})'$ is the mean response set for variable k. Following the earlier development of the structured latent curve model for a single variable, formulation of the multivariate model begins by specifying the mean forms of the individual variables. There is no restriction that the means of the different variables follow the same form. For example, the means of one variable may be assumed to follow an exponential function and the means of another a Gompertz function. The multivariate mean vector then is assumed to follow a target function vector:

$$\begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \\ \vdots \\ \boldsymbol{\mu}_K \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1(\boldsymbol{\theta}_1, \mathbf{t}) \\ \mathbf{f}_2(\theta_2, \mathbf{t}) \\ \vdots \\ \mathbf{f}_K(\theta_K, \mathbf{t}) \end{bmatrix},$$

where $\mathbf{f}_k(\boldsymbol{\theta}_k, \mathbf{t})$ is the target function for variable k.

The structured latent curve model for the multivariate response set is formed by stacking the submodels of the individual response variables (Goldstein, 1995; MacCallum et al., 1997). Assuming that each response variable follows a structured latent curve model, the model for the multivariate set of responses, $\mathbf{y}_i$, is

$$\begin{bmatrix} \mathbf{y}_{1i} \\ \mathbf{y}_{2i} \\ \vdots \\ \mathbf{y}_{Ki} \end{bmatrix} = \begin{bmatrix} \mathbf{\Lambda}_{1i} & & & \\ & \mathbf{\Lambda}_{2i} & & \\ & & \ddots & \\ & & & \mathbf{\Lambda}_{Ki} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta}_{1i} \\ \boldsymbol{\eta}_{2i} \\ \vdots \\ \boldsymbol{\eta}_{Ki} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\epsilon}_{1i} \\ \boldsymbol{\epsilon}_{2i} \\ \vdots \\ \boldsymbol{\epsilon}_{Ki} \end{bmatrix}$$

$$\mathbf{y}_i = \mathbf{\Lambda}_i \boldsymbol{\eta}_i + \boldsymbol{\epsilon}_i. \tag{9}$$

The factor matrix, $\mathbf{\Lambda}_i$, is block diagonal. The diagonal block elements are the factor loading matrices of the individual response models, for $k = 1, \ldots, K$. The matrix has rows equal to the total number of observations across all variables for individual i and columns equal to the total number of basis curves for all K response variables. The vector $\boldsymbol{\eta}_i$ is the multivariate random coefficient vector containing individual i's entire set of random coefficients for all K variables: $\boldsymbol{\eta}_i = (\boldsymbol{\eta}'_{1i}, \boldsymbol{\eta}'_{2i}, \ldots, \boldsymbol{\eta}'_{Ki})'$; $\boldsymbol{\eta}_i$ is of length J, where J is the total number of basis curves across all response variables. The random coefficients correspond to individual-level characteristics of change in the different response variables. The vector $\boldsymbol{\epsilon}_i$ is a multivariate vector of measurement errors representing the individual-level discrepancies between the observed values $\mathbf{y}_i$ and the latent curves given by $\mathbf{\Lambda}_i\boldsymbol{\eta}_i$. The errors are assumed to be independent of the random coefficients and to be normally distributed with zero means and a covariance matrix $\mathbf{\Theta}_{\epsilon i}$:

$$\begin{bmatrix} \boldsymbol{\epsilon}_{1i} \\ \boldsymbol{\epsilon}_{2i} \\ \vdots \\ \boldsymbol{\epsilon}_{Ki} \end{bmatrix} \sim N\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{\Theta}_{\epsilon 1} & & & \\ \mathbf{\Theta}_{\epsilon 2 \epsilon 1} & \mathbf{\Theta}_{\epsilon 2} & & \\ \vdots & \vdots & \ddots & \\ \mathbf{\Theta}_{\epsilon K \epsilon 1} & \mathbf{\Theta}_{\epsilon K \epsilon 2} & \cdots & \mathbf{\Theta}_{\epsilon K} \end{bmatrix} \right). \tag{10}$$

The diagonal submatrices of $\mathbf{\Theta}_{\epsilon i}$, $\mathbf{\Theta}_{\epsilon k}$, are the error covariance matrices of the individual response variables. The off-diagonal submatrices, $\mathbf{\Theta}_{\epsilon l \epsilon k}$, are not symmetric covariance matrices but rather contain the covariances between the measurement errors of different variables, for $1 \leq l, k \leq K$, $l \neq k$. In many settings, it may be reasonable to assume that measurement errors between variables are independent, in which case $\mathbf{\Theta}_{\epsilon i}$ would be a block diagonal matrix:

$$\mathbf{\Theta}_{\epsilon i} = \begin{bmatrix} \mathbf{\Theta}_{\epsilon 1} & & & \\ & \mathbf{\Theta}_{\epsilon 2} & & \\ & & \ddots & \\ & & & \mathbf{\Theta}_{\epsilon K} \end{bmatrix}.$$

At the individual level, the random coefficients are assumed to be multivariate normal with mean vector $\boldsymbol{\alpha}$ and covariance matrix $\mathbf{\Psi}$, where

$$\boldsymbol{\alpha} = E\begin{bmatrix} \boldsymbol{\eta}_{1i} \\ \boldsymbol{\eta}_{2i} \\ \vdots \\ \boldsymbol{\eta}_{Ki} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \vdots \\ \boldsymbol{\alpha}_K \end{bmatrix} \tag{11a}$$

and

$$\mathbf{\Psi} = cov(\boldsymbol{\eta}_i) = \begin{bmatrix} \mathbf{\Psi}_1 & & & \\ \mathbf{\Psi}_{21} & \mathbf{\Psi}_2 & & \\ \vdots & \vdots & \ddots & \\ \mathbf{\Psi}_{K1} & \mathbf{\Psi}_{K2} & \cdots & \mathbf{\Psi}_K \end{bmatrix}. \tag{11b}$$

The matrix $\mathbf{\Psi}$ is a symmetric block covariance matrix. The diagonal blocks of $\mathbf{\Psi}$ represent symmetric covariance matrices of the random coefficients for the individual response variables: $\mathbf{\Psi}_{kk} = cov(\boldsymbol{\eta}_{ki}, \boldsymbol{\eta}'_{ki})$, $k = 1, \ldots, K$. These covariance matrices represent information about individual differences in response variables considered separately. The off-diagonal blocks are matrices containing the covariances between the random coefficients of different response variables: $\mathbf{\Psi}_{lk} = cov(\boldsymbol{\eta}_{li}, \boldsymbol{\eta}'_{ki})$, $1 \leq l, k \leq K$, $l \neq k$. These submatrices represent the linear associations between characteristics of change in the different response variables and, thus, represent the added information of an analysis that considers multiple response variables jointly. It is probably most common to assume that $\mathbf{\Psi}$ is unstructured to allow for estimation of all covariances between random coefficients. The model in Equation 9 and the assumptions in Equations 10, 11a, and 11b give the marginal mean vector and covariance matrix of $\mathbf{y}_i$:

$$\boldsymbol{\mu}_i = \begin{bmatrix} \mathbf{\Lambda}_{1i} & & & \\ & \mathbf{\Lambda}_{2i} & & \\ & & \ddots & \\ & & & \mathbf{\Lambda}_{Ki} \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \vdots \\ \boldsymbol{\alpha}_K \end{bmatrix}$$

$$\mathbf{\Sigma}_i = \mathbf{\Lambda}_i \mathbf{\Psi} \mathbf{\Lambda}'_i + \mathbf{\Theta}_{\epsilon i}.$$

Factor Analysis Model for Covariates

A factor analysis model for a separate set of covariates is presented here. This component of the model is appropriate for a set of normal variables considered to be related to one or more latent variables. Such variables may be those whose values are assumed to be stable over the course during which the repeated measures are observed, or possibly observed at one point in time, such as baseline measures of covariates. In a factor analysis model, it is assumed that the set of manifest variables, denoted here by $\mathbf{y}_c$, is a linear function of the common and unique factors (cf. Lawley & Maxwell, 1971):

$$\mathbf{y}_c = \mathbf{\Lambda}_c \boldsymbol{\xi}_i + \boldsymbol{\delta}_i,$$

where $\mathbf{\Lambda}_c$ is a $P \times Q$ factor matrix with loadings $\boldsymbol{\lambda}_c = (\lambda_1, \ldots)'$, $\boldsymbol{\xi}_i$ is a $Q \times 1$ common factor vector, and $\boldsymbol{\delta}_i$ is a $Q \times 1$ vector of uniquenesses. The factor has mean $\boldsymbol{\mu}_\xi$ and covariance matrix $\mathbf{\Phi}_\xi$. It is assumed that the unique factors are normally distributed with mean zero and covariance matrix $\mathbf{\Theta}_\delta$. The covariances among the uniquenesses are assumed to be zero, but the variances may differ between variables. The variances of the uniquenesses are also assumed to be nonnegative and uncorrelated with the factors. Under the model, it is assumed that the linear associations among the observed variables are accounted for by the latent variables (Browne, 1982).

It is necessary to set constraints on model parameters to help remove the indeterminacy of a model with no such restrictions. This is typically done by assuming either that the common factors have unit variance or that the scale on which a factor is measured is equal to that of one of its measured variables. Such restrictions, however, do not ensure identifiability of the model (see Lawley & Maxwell, 1971). The preceding model assumptions imply that $\mathbf{y}_c$ is normal with expected value and covariance matrix

$$\boldsymbol{\mu}_{\mathbf{c}} = \boldsymbol{\Lambda}_c \boldsymbol{\mu}_\xi$$

and

$$\boldsymbol{\Sigma}_{\mathbf{c}} = \boldsymbol{\Lambda}_c \boldsymbol{\Phi}_\xi \boldsymbol{\Lambda}_c' + \boldsymbol{\Theta}_\delta.$$

The factor analysis model for covariates also allows for missing data in the set of observed variables (Finkbiner, 1979). This may be handled by creating an index vector with elements corresponding to the observed values of $\mathbf{y}_c$. Let $\mathbf{k} = [1, 2, \ldots, P]'$. For each individual, the pattern of observations in $\mathbf{y}_c$ is specified by $\mathbf{k}_i$. For example, assuming a model for three observed covariates, individual i with responses to the first and third manifest variables would have $\mathbf{k}_i = [1, 3]'$. The individual's data vector would have data values corresponding to $[\mathbf{y}_c]_{\mathbf{k}}$. Let $\mathbf{y}_{ci}$ denote an individual's data vector that may be incomplete. The factor analysis model that allows for missing data follows as

$$\mathbf{y}_{ci} = [\boldsymbol{\Lambda}_c \boldsymbol{\xi}_i + \boldsymbol{\delta}_i]_{\mathbf{k}} = \boldsymbol{\Lambda}_{ci} \boldsymbol{\xi}_i + \boldsymbol{\delta}_i,$$

where $\boldsymbol{\Lambda}_{ci}$ and $\boldsymbol{\delta}_i$ have rows corresponding to observations in $\mathbf{y}_{ci}$. Allowing for individuals to have different observed data patterns in the covariates, the mean vector and covariance matrix of $\mathbf{y}_c$ are

$$\boldsymbol{\mu}_{\mathbf{c}_i} = \boldsymbol{\Lambda}_{ci} \boldsymbol{\mu}_\xi$$

and

$$\boldsymbol{\Sigma}_{\mathbf{c}_i} = \boldsymbol{\Lambda}_{ci} \boldsymbol{\Phi}_\xi \boldsymbol{\Lambda}_{ci}' + \boldsymbol{\Theta}_{\delta i}.$$

The order of the mean vector $\boldsymbol{\mu}_{\mathbf{c}_i}$ and the covariance matrix $\boldsymbol{\Sigma}_{\mathbf{c}_i}$ of the manifest covariates is P_i, where P_i is the number of manifest variables observed for individual i.

The Full Model

Following Browne (1993), the structured latent curve model for multiple normal repeated measures is combined with a factor analysis submodel for covariates. The full model allows for estimation of the covariances between the random coefficients in the second level of the structured latent curve model and a set of latent covariates. Changing earlier notation, let $\mathbf{y}_{Ri}$ denote the response vector corresponding to the set of repeated measures variables, with $\mathbf{y}_{ci}$ again being the response vector corresponding to the covariates. Similar to the steps taken to form the multivariate response vector for the repeated measures, the responses to both the repeated measures and the covariates are stacked to form a single response vector: $\mathbf{y}_i = (\mathbf{y}_{Ri}', \mathbf{y}_{ci}')'$, where $\mathbf{y}_i$ now represents a stacked vector of observed scores for the ith case. The full model is

$$\begin{bmatrix} \mathbf{y}_{Ri} \\ \mathbf{y}_{ci} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Lambda}_{Ri} & \\ & \boldsymbol{\Lambda}_{ci} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta}_i \\ \boldsymbol{\xi}_i \end{bmatrix} + \begin{bmatrix} \boldsymbol{\epsilon}_i \\ \boldsymbol{\delta}_i \end{bmatrix}$$

$$\mathbf{y}_i = \boldsymbol{\Lambda}_i \boldsymbol{\varphi}_i + \boldsymbol{\omega}_i,$$

where $\mathbf{y}_i$ is a response vector whose length is given by the number of responses across the repeated measures variables (T_i) and the number of responses to the set of covariates (P_i). The factor loading matrix $\boldsymbol{\Lambda}_i$ is partly a function of the parameters characterizing change in the mean response and the times of measurement corresponding to the repeated measures, that is, $\boldsymbol{\Lambda}_{Ri} = \boldsymbol{\Lambda}_{Ri}(\boldsymbol{\theta}, \mathbf{t}_i)$, and the factor loadings corresponding to the factor analysis model for the covariates, that is, $\boldsymbol{\Lambda}_{ci} = \boldsymbol{\Lambda}_{ci}(\boldsymbol{\lambda})$. The matrix $\boldsymbol{\Lambda}_i$ includes the subscript i so that, for individual i, its rows correspond to the number and actual times of measurement by which the repeated measures are observed as well as the number of covariates observed; conversely, the number of columns in $\boldsymbol{\Lambda}_i$ is the same across all individuals. The coefficient vector $\boldsymbol{\varphi}_i$ is a stacked $(J + Q)$ vector of the random coefficients and the factors. The variate $\boldsymbol{\omega}_i$ is a stacked $(T_i + P_i)$ vector of measurement errors corresponding to the structured latent curve model for the repeated measures and the factor analysis model for the covariates. Assuming that the expected value of $\boldsymbol{\varphi}_i$ is $E(\boldsymbol{\varphi}_i) = \boldsymbol{\varphi}$, the mean vector and covariance matrix of $\mathbf{y}_i$ are

$$\boldsymbol{\mu}_i = \boldsymbol{\Lambda}_i \boldsymbol{\varphi}$$

and

$$\boldsymbol{\Sigma}_i = \boldsymbol{\Lambda}_i \boldsymbol{\Phi} \boldsymbol{\Lambda}_i' + \boldsymbol{\Theta}_i,$$

where $\boldsymbol{\Phi}$ is a symmetric block covariance matrix of the random coefficients of the multivariate structured latent curve model and the latent covariates:

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\Psi} & \\ \boldsymbol{\Phi}_{\xi\eta} & \boldsymbol{\Phi}_\xi \end{bmatrix}.$$

Block elements of $\boldsymbol{\Phi}$ characterize the linear associations among the random coefficients of the latent curve model ($\boldsymbol{\Psi}$), the latent covariates ($\boldsymbol{\Phi}_\xi$), and the associations between the stochastic coefficients of the two submodels ($\boldsymbol{\Phi}_{\xi\eta}$). That is, $\boldsymbol{\Psi}$ contains the variances and covariances of the different characteristics of change in the repeated measures. The matrix $\mathbf{\Phi}_{\xi}$ contains the variances and covariances of the latent covariates. The matrix $\mathbf{\Phi}_{\xi\eta}$ contains the covariances between the characteristics of change in the set of repeated measures and the latent covariates. This latter matrix provides the information gained by treating the repeated measures and the latent covariates simultaneously. Finally, variation within individuals is summarized in the covariance matrix $\mathbf{\Theta}_i$, which is assumed to be block diagonal such that the measurement errors of the repeated measures and the uniquenesses of the covariates are independent. For cases of unbalanced data, the covariance matrix of the measurement errors is

$$\mathbf{\Theta}_i = \begin{bmatrix} \mathbf{\Theta}_{ei} & \\ & \mathbf{\Theta}_{\delta i} \end{bmatrix},$$

where the rows of each submatrix correspond to the pattern of observed responses for individual *i*. The matrices $\mathbf{\Phi}$ and $\mathbf{\Theta}_i$ that characterize between-individual and within-individual variability, respectively, are taken to be positive–definite.

Missing Data

Missing data on either the repeated measures or the covariates are possible. Within individuals, the set of repeated measures may be observed according to different time points or for a different number of occasions, either of which may occur by design or possibly by attrition, for example. Between individuals, the repeated measures may also be observed at different time points or for a different number of occasions. Generally, the model presented allows for different patterns of observed data both within and between individuals. In the case of missing data, the model may be valid when the reason for the missing data is ignorable (Little & Rubin, 1987; Rubin, 1976), and thus the data are missing at random. That is, valid inference is based on the assumption that any missing data are independent of the reason for their absence (Laird, 1988). In cases of data not missing at random, a random-coefficient pattern mixture model may be considered (Hedeker & Gibbons, 1997).

Estimation

Standard software packages for the estimation of structural equation models, such as LISREL, AMOS, and Mplus, may be used for estimation of a random coefficient model that may also include a factor analysis model for covariates. When data are complete and individuals are observed according to the same data collection scheme (even if intervals are unequally spaced), software programs that implement an estimation procedure based on sufficient statistics, such as a sample mean vector and covariance matrix, may be used to estimate the model. Willett and Sayer (1994) provided details for fitting a latent curve model with LISREL. For cases in which the repeated measures are balanced with respect to time but some observations of either the repeated measures or the covariates are missing, software programs implementing estimation procedures based on raw data, such as LISREL Version 8.5, which implements full information maximum likelihood estimation, may be used to estimate the model. When the repeated measures are observed according to different time points, however, software that allows for the factor matrix, $\mathbf{\Lambda}_i$, to vary across individuals is needed. In these latter cases, software programs such as SAS (e.g., PROC NLMIXED) and Mx (Neale, Boker, Xie, & Maes, 2002) may be used. Fitting structured latent curve models typically involves constraints on the factor loading matrix to allow for some parameters to enter nonlinearly. Depending in part on how the data are collected (i.e., whether the data are complete or balanced with respect to times of measurement), some forms of the structured latent curve model may be fitted with standard software, such as LISREL, whereas others may require specialized software, such as Mx.

Example

Data from a learning study were used to illustrate the method.[1] The data represent performance on two procedural tasks developed for the assessment of quantitative and verbal skill acquisition. For each task, participants were required to learn a set of declarative rules for evaluating characteristics of visual stimuli presented in series. Tasks were given together in blocks, with a mixed order within blocks. Both response times and accuracy scores were recorded for a total of 384 trials for each task. Data for 228 individuals whose average accuracy score across trial blocks was 80% or better on both tasks are considered here. Restricting the sample to individuals with higher accuracy levels was done to reduce any influence of a speed–accuracy trade-off on response time scores. Response times for each task were aggregated separately into 12 blocks of 32 trials each. Aggregated data represent the median time to respond within blocks. In addition to the procedural learning tasks, participants were given a battery of tasks designed to measure working memory (WM). The set of WM measures was hypothesized to follow a factor analysis model such that a single latent measure of WM was assumed to give rise to the observed task scores, with provisions made for the errors of measurement contained in the observed scores. A subset of 204 individuals had complete data for the WM battery. The goals of the present analysis were to describe (a) the mean response times on the quantitative and verbal procedural

[1] The data were provided by Scott Chaiken of the Armstrong Laboratory, Brooks Air Force Base.

tasks separately, (b) the variation and covariation among individual-level characteristics of change in response times on the two tasks, and (c) the covariation among the individual-level characteristics of change in response times and a latent measure of WM to assess the role of WM in procedural learning. One approach to including WM as a correlate of procedural learning is to create a composite variable by taking a weighted sum of scores from the individual tests. Browne and du Toit (1991) incorporated a similarly created composite variable into a structured latent curve model for responses to a single learning task. In their model, the composite variable, assumed to be measured without error, is considered a correlate of the individual-level weights. Browne (1993) then considered a latent variable model for covariates and incorporated this into a structured latent curve model for a single learning variable.

Procedural Learning Tasks

Let $\mathbf{y}_{Qi}$ and $\mathbf{y}_{Vi}$ denote the sets of response times on the quantitative (Q) and verbal (V) procedural tasks, respectively, for individual i, $i = 1, \ldots, 228$. For each task, the set of responses $\mathbf{y}_{ki}$ contained median response times for 12 blocks, that is, $\mathbf{t} = (1, 2, \ldots, 12)'$ for task k, $k = Q, V$. In all cases, data for both tasks were complete and balanced with respect to trial blocks. Figure 1 shows a 10% subsample of responses on each task, separately, across trial blocks. Observed mean responses on the two tasks are shown separately in Figure 2. As can be seen in Figure 2, mean responses on both tasks were characterized by an initial rapid decrease followed by a slow, decreasing trend that leveled off during the final trials. A negatively accelerated exponential function decreasing monotonically to an asymptote that was greater than zero seemed appropriate for each mean response set (Meredith & Tisak, 1990, p. 117):

$$\mu_k = f(\boldsymbol{\theta}_k, t) = \theta_{1k} - (\theta_{1k} - \theta_{2k})\, exp[-\theta_{3k}(t-1)] + \epsilon_{tki}, \qquad k = Q, V,$$

where, for task k, θ_{1k} and θ_{2k} represent potential and initial response times, respectively, and θ_{3k} is the population initial rate of change. The parameters of the model are interpreted with regard to different aspects of the learning process. At the first trial block, the function value is equal to θ_{2k}, the coefficient representing response time on the task at the start of the series. As trial blocks approach infinity, the function value is equal to θ_{1k}, the true response time at the lower asymptote, thus representing potential performance. Finally, θ_{3k} represents the initial rate of change in response time. Considering performance on the two procedural tasks simultaneously, a structured latent curve model for $\mathbf{y}_i = (\mathbf{y}'_{Qi}, \mathbf{y}'_{Vi})'$ was

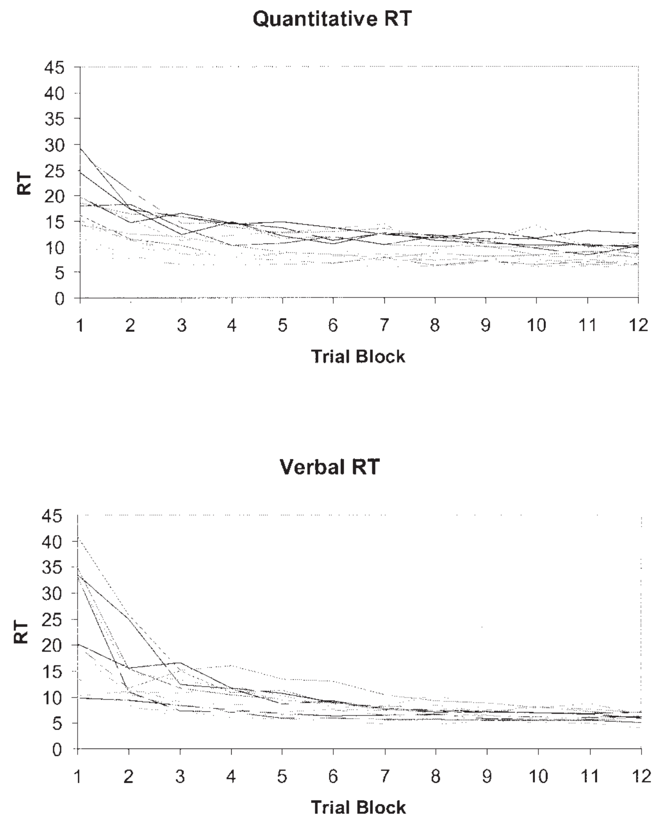


Figure 1. Subsample (10%) of repeated response time (RT) scores on the quantitative (top) and verbal (bottom) procedural learning tasks.

$$\begin{bmatrix} \mathbf{y}_{Qi} \\ \mathbf{y}_{Vi} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Lambda}_Q & \\ & \boldsymbol{\Lambda}_V \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta}_{Qi} \\ \boldsymbol{\eta}_{Vi} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\epsilon}_{Qi} \\ \boldsymbol{\epsilon}_{Vi} \end{bmatrix}$$

$$\mathbf{y}_i = \boldsymbol{\Lambda}_R \boldsymbol{\eta}_i + \boldsymbol{\epsilon}_i,$$

where $\boldsymbol{\Lambda}_R$ was a (24×6) block diagonal matrix containing basis curves for both procedural learning tasks, $\boldsymbol{\eta}_i$ was a (6×1) stacked vector of random coefficients for the two tasks, and $\boldsymbol{\epsilon}_i$ was a (24×1) error variate. For the kth procedural learning task, the target function and its corresponding three basis curves for the exponential function were

Target function:

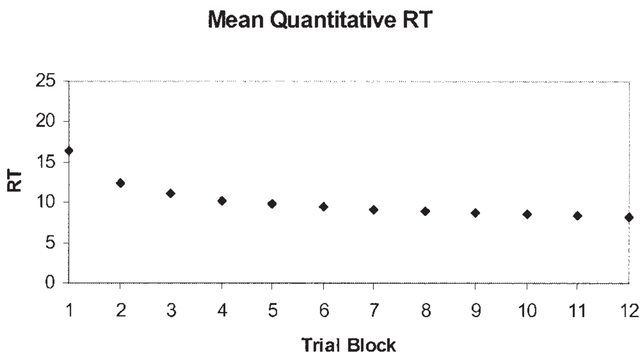
$$f(\boldsymbol{\theta}_k, t) = \theta_{1k} - (\theta_{1k} - \theta_{2k})\, exp[-\theta_{3k}(t-1)]$$

Lower response time asymptote basis curve:

$$f'_1(\boldsymbol{\theta}_k, t) = \frac{\partial f(\boldsymbol{\theta}_k, t)}{\partial \theta_{1k}} = 1 - exp[-\theta_{3k}(t-1)]$$

Initial response time basis curve:

$$f'_2(\boldsymbol{\theta}_k, t) = \frac{\partial f(\boldsymbol{\theta}_k, t)}{\partial \theta_{2k}} = exp[-\theta_{3k}(t-1)]$$



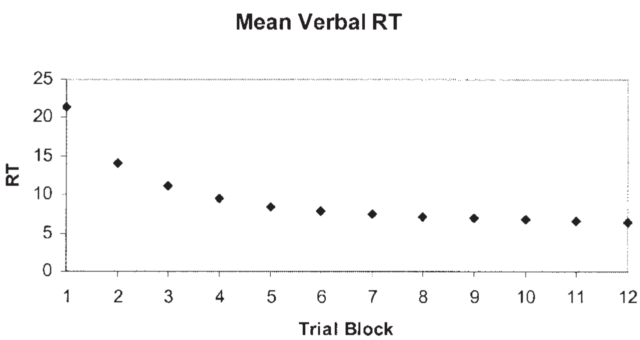


Figure 2. Observed trial block means on the quantitative (top) and verbal (bottom) procedural tasks. RT = response time.

Initial rate of change in response time basis curve:

$$f_3'(\boldsymbol{\theta}_k, t) = \frac{\partial f(\boldsymbol{\theta}_k, t)}{\partial \theta_{3k}} = (\theta_{1k} - \theta_{2k})(t-1)\, exp[-\theta_{3k}(t-1)].$$

In the current application, the factor matrix $\boldsymbol{\Lambda}_R$ was block diagonal with six columns:

$$\boldsymbol{\Lambda}_\mathbf{R} = \begin{bmatrix} f_{1Q}'(1, \boldsymbol{\theta}_Q) & f_{2Q}'(1, \boldsymbol{\theta}_Q) & f_{3Q}'(1, \boldsymbol{\theta}_Q) & 0 & 0 & 0 \\ f_{1Q}'(2, \boldsymbol{\theta}_Q) & f_{2Q}'(2, \boldsymbol{\theta}_Q) & f_{3Q}'(2, \boldsymbol{\theta}_Q) & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{1Q}'(12, \boldsymbol{\theta}_Q) & f_{2Q}'(12, \boldsymbol{\theta}_Q) & f_{3Q}'(12, \boldsymbol{\theta}_Q) & 0 & 0 & 0 \\ 0 & 0 & 0 & f_{1V}'(1, \boldsymbol{\theta}_V) & f_{2V}'(1, \boldsymbol{\theta}_V) & f_{3V}'(1, \boldsymbol{\theta}_V) \\ 0 & 0 & 0 & f_{1V}'(2, \boldsymbol{\theta}_V) & f_{2V}'(2, \boldsymbol{\theta}_V) & f_{3V}'(2, \boldsymbol{\theta}_V) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & f_{1V}'(12, \boldsymbol{\theta}_V) & f_{2V}'(12, \boldsymbol{\theta}_V) & f_{3V}'(12, \boldsymbol{\theta}_V) \end{bmatrix},$$

where $\boldsymbol{\theta}_Q = (\theta_{Q1}, \theta_{Q2}, \theta_{Q3})'$ and $\boldsymbol{\theta}_V = (\theta_{V1}, \theta_{V2}, \theta_{V3})'$ were the parameters of the response time target functions for the quantitative and verbal tasks, respectively. The random coefficients in $\boldsymbol{\eta}_i$ have substantive interpretations analogous to the fixed coefficients of the target functions. The two differ in that coefficients of the target functions represent change characteristics for the mean response, and the random coefficients represent change characteristics particular to each individual. Although the expected values of the common scores were assumed to follow the monotonic functional form of $f(\boldsymbol{\theta}, t)$, the common scores themselves were not.

Two competing structures for the within-individual error matrix were examined, considering each task separately. In one model, within-subject errors were assumed to be independent and normally distributed with a mean of zero and constant variance across trial blocks: $\boldsymbol{\epsilon}_{ki} \sim N(\mathbf{0}, \sigma_k^2\mathbf{I})$, where $\mathbf{I}$ was a 12 × 12 identity matrix and σ_k^2 was a variance coefficient for task k, $k = Q, V$. In a competing model, the errors were assumed to have an autoregressive structure to allow for a decreasing interdependence of within-subject errors between trial blocks as distances between blocks increased:

$$\sigma_k^2 \rho_k^{(r-s)},$$

where σ_k^2 is the error variance and ρ_k is the autocorrelation coefficient, with r and s denoting the rows and columns of the covariance matrix $\boldsymbol{\Theta}_k$, $k = Q, V$ (Browne, 1993). Treating responses to each procedural task separately, the two models distinguished by the different error structures were compared by calculating the Akaike information criterion (AIC). This index of model fit takes into account the number of parameters, penalizing models with greater numbers. The AIC may be defined as

$$AIC_j = -2 \ln L + 2q_j,$$

where ln L is the natural log of the likelihood function value and q is the number of parameters in the model. In this form of the index, the model yielding the smallest value is preferred. On the basis of a comparison of AIC values, the model that included the autoregressive error structure for within-individual variation seemed preferable for both outcomes and was provisionally accepted.[2] Considering the tasks together, the within-individual errors between tasks were assumed to be independent. That is, $\boldsymbol{\Theta}_\epsilon$ was block diagonal:

$$\boldsymbol{\Theta}_\epsilon = \begin{bmatrix} \boldsymbol{\Theta}_Q & \\ & \boldsymbol{\Theta}_V \end{bmatrix}.$$

The distribution of the random coefficients was assumed to be

$$\boldsymbol{\eta}_i \sim N(\boldsymbol{\alpha}, \boldsymbol{\Psi}),$$

[2] For the quantitative responses, the AIC fit index values were 4,845.3 (assuming independent and constant error variance) and 4,802.9 (assuming an autoregressive error structure). For the verbal responses, the AIC fit index values were 6,533.9 (assuming independent and constant error variance) and 6,427.9 (assuming an autoregressive error structure).

where $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_Q, \boldsymbol{\alpha}_V)$ was the vector of response time factor means. The expected values of the factor means were obtained by solving the set of linear equations: $f_k(\boldsymbol{\theta}_k, \mathbf{t}) = \boldsymbol{\Lambda}_k\boldsymbol{\alpha}_k$. Here $\boldsymbol{\alpha}_Q = (\theta_{Q1}, \theta_{Q2}, 0)'$ and $\boldsymbol{\alpha}_V = (\theta_{V1}, \theta_{V2}, 0)'$. The covariances among the random coefficients were given by $\boldsymbol{\Psi}$:

$$\boldsymbol{\Psi} = \begin{bmatrix} \boldsymbol{\Psi}_Q & \\ \boldsymbol{\Psi}_{VQ} & \boldsymbol{\Psi}_V \end{bmatrix}.$$

The matrices $\boldsymbol{\Psi}_Q$ and $\boldsymbol{\Psi}_V$ were symmetric covariance matrices for the random coefficients of the two learning models for quantitative and verbal performance, respectively. The off-diagonal matrix, $\boldsymbol{\Psi}_{VQ}$, was a nonsymmetric matrix containing the covariances between the random coefficients of the two response time models:

$$\boldsymbol{\Psi}_{VQ} = \begin{bmatrix} cov(\eta_{V1}, \eta_{Q1}) & cov(\eta_{V1}, \eta_{Q2}) & cov(\eta_{V1}, \eta_{Q3}) \\ cov(\eta_{V2}, \eta_{Q1}) & cov(\eta_{V2}, \eta_{Q2}) & cov(\eta_{V2}, \eta_{Q3}) \\ cov(\eta_{V3}, \eta_{Q1}) & cov(\eta_{V3}, \eta_{Q2}) & cov(\eta_{V3}, \eta_{Q3}) \end{bmatrix}.$$

The matrix $\boldsymbol{\Psi}_{VQ}$ represented the linear associations between the random coefficients of the two models obtained by considering the two tasks simultaneously. Given assumptions about the within-individual and between-individual sources of variation, the mean vector and covariance matrix of $\mathbf{y}_i$ were

$$\boldsymbol{\mu}_i = \boldsymbol{\Lambda}_{Ri}\boldsymbol{\alpha}$$

and

$$\boldsymbol{\Sigma}_i = \boldsymbol{\Lambda}_R\boldsymbol{\Psi}\boldsymbol{\Lambda}_R' + \boldsymbol{\Theta}_\epsilon.$$

Working Memory Battery

On a separate occasion, a subset of 204 participants were administered a battery of tasks designed to measure WM (see Kyllonen & Christal, 1990). Four tests were considered in which two, q_1 and q_2, were quantitative in nature and two others, v_1 and v_2, were verbal in nature. The four observed measures of WM were assumed to follow a factor analysis model. Let $\mathbf{c}_i = (q_{1i}, q_{2i}, v_{1i}, v_{2i})'$ represent the set of responses for individual i on the four tasks. It was assumed that $\mathbf{c}_i$ followed a one-factor model:

$$\begin{bmatrix} q_{1i} \\ q_{2i} \\ v_{1i} \\ v_{2i} \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} [\xi_i] + \begin{bmatrix} \delta_{1i} \\ \delta_{2i} \\ \delta_{3i} \\ \delta_{4i} \end{bmatrix}$$

$$\mathbf{c}_i = \lambda\xi_i + \delta_i,$$

where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_4)'$ was a vector of factor loadings and ξ_i was a latent measure of WM. The vector $\boldsymbol{\delta}_i = (\delta_{1i}, \ldots, \delta_{4i})'$ was the set of measurement errors (i.e., uniquenesses) resulting from the regression of $\mathbf{c}_i$ on ξ_i. It was assumed that the mean of ξ_i was μ_ξ. The variance of ξ_i was set equal to unity: $var(\xi_i) = 1$. The errors were assumed to be normally distributed with zero means and symmetric variance and covariance matrix $\boldsymbol{\Theta}_\delta$; the uniquenesses were assumed to be independent with possibly different variances:

$$\boldsymbol{\delta}_i \sim N(\mathbf{0}, \boldsymbol{\Theta}_\delta),$$

where

$$\boldsymbol{\Theta}_\delta = \begin{bmatrix} \sigma^2_{\delta_1} & & & \\ & \sigma^2_{\delta_2} & & \\ & & \sigma^2_{\delta_3} & \\ & & & \sigma^2_{\delta_4} \end{bmatrix}.$$

The error variances were assumed to be nonnegative, and the errors were assumed to be independent of the factor. With these assumptions, the expected value and covariance matrix of the observed covariates were

$$\boldsymbol{\mu}_c = \boldsymbol{\lambda}\boldsymbol{\mu}_\xi$$

and

$$\boldsymbol{\Sigma}_c = \boldsymbol{\lambda}\boldsymbol{\lambda}' + \boldsymbol{\Theta}_\delta.$$

The one-factor model was fitted to scores from the WM task battery using maximum likelihood estimation, $\chi^2(5, N = 204) = 6.72$. The sample correlation coefficients ranged from .28 to .45. The maximum absolute value of the discrepancies between the fitted and sample correlations was less than .05, suggesting that the sample correlations were reasonably accounted for by a single factor. Model fit was further evaluated by calculating the root mean square error of approximation (RMSEA; Steiger, 1990). The factor analysis model for the battery of measures yielded an RMSEA value of .041, suggesting a reasonable fit to the data (Browne & Cudeck, 1992).

The Full Model

The models for response time scores on the two procedural learning tasks and the factor analysis model for the WM battery were incorporated into a single model. Let $\mathbf{y}_i = (\mathbf{y}_{Qi}', \mathbf{y}_{Vi}', \mathbf{y}_{ci}')'$ be the response vector for individual i with complete data on all variables, composed by stacking the individual response sets of quantitative and verbal response times and WM task scores. A subset of 24 individuals were missing data for the WM task battery. The tasks examined here represented a subset of tasks required for a much larger research project. Some individuals were unable to complete all components of the project as a result of the large number of tasks administered and the time necessary to complete them. For this analysis, it seemed reasonable to assume that the missing data on the WM task battery were missing at random. Specifically, there was no indication that the reasons for the missing data were related to how individuals would have performed had they been administered the battery. Therefore, the data were considered to be missing at random in the sense that the missing data mechanisms were ignorable (Little & Rubin, 1987; Rubin, 1976). For individuals missing the battery of WM tasks, the response set was composed of scores from only the procedural learning tasks, that is, $\mathbf{y}_i = (\mathbf{y}'_{Qi}, \mathbf{y}'_{Vi})'$. Thus, the number of covariates was equal to P_i, where $P_i = 4$ for individuals who completed the WM battery and $P_i = 0$ for those who did not.

Considering simultaneously the responses to the procedural learning tasks and the WM battery, the model for $\mathbf{y}_i$ was assumed to be

$$\begin{bmatrix} \mathbf{y}_{Qi} \\ \mathbf{y}_{Vi} \\ \mathbf{y}_{ci} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Lambda}_Q & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda}_V & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Lambda}_c \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta}_{Qi} \\ \boldsymbol{\eta}_{Vi} \\ \boldsymbol{\xi}_i \end{bmatrix} + \begin{bmatrix} \boldsymbol{\epsilon}_{Qi} \\ \boldsymbol{\epsilon}_{Vi} \\ \boldsymbol{\delta}_i \end{bmatrix}$$

$$\mathbf{y}_i = \boldsymbol{\Lambda}_i \boldsymbol{\varphi}_i + \boldsymbol{\omega}_i, \tag{12}$$

where the factor matrix $\boldsymbol{\Lambda}_i = \boldsymbol{\Lambda}_i(\boldsymbol{\theta}', \mathbf{t}', \boldsymbol{\lambda}')$ was block diagonal and the coefficient vector $\boldsymbol{\varphi}_i = (\boldsymbol{\eta}'_{Qi}, \boldsymbol{\eta}'_{Vi}, \boldsymbol{\xi}_i)'$ contained the random coefficients for the two procedural learning models and the WM factor. Within individuals, the errors were assumed to have distribution

$$\boldsymbol{\omega}_i \sim N(\mathbf{0}, \boldsymbol{\Theta}_i), \tag{13}$$

where $\boldsymbol{\Theta}_i$ was of order $(24 + P_i)$. The matrix $\boldsymbol{\Theta}_i$ was assumed to be block diagonal so that errors between tasks were independent:

$$\boldsymbol{\Theta}_i = \begin{bmatrix} \boldsymbol{\Theta}_Q & & \\ & \boldsymbol{\Theta}_V & \\ & & \boldsymbol{\Theta}_\delta \end{bmatrix}.$$

The error matrix $\boldsymbol{\Theta}_i$ was assumed to be homogeneous across individuals but could vary with regard to order to handle missing data in the WM battery. That is, for individuals with complete data, $\boldsymbol{\Theta}_i$ had rows corresponding to the 24 responses of the repeated measures variables along with the four rows corresponding to the set of WM task measures. For individuals with incomplete data, $\boldsymbol{\Theta}_i$ had rows corresponding to the 24 responses on the procedural tasks. Within individuals, the measurement errors were assumed to be independent of the random coefficients.

The random coefficients and the latent covariates were assumed to have distribution

$$\boldsymbol{\varphi}_i \sim N(\boldsymbol{\mu}_{\varphi}, \boldsymbol{\Phi}), \tag{14}$$

where $\boldsymbol{\mu}_{\varphi} = (\boldsymbol{\alpha}'_Q, \boldsymbol{\alpha}'_V, \mu_\xi)'$. The covariance matrix among these coefficients was represented by $\boldsymbol{\Phi}$, where $\boldsymbol{\Phi}$ was a symmetric block covariance matrix of the random coefficients of the two procedural tasks and the latent WM variable. Specifically,

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\Psi}_Q & & \\ \boldsymbol{\Psi}_{VQ} & \boldsymbol{\Psi}_V & \\ \boldsymbol{\Phi}_{\xi Q} & \boldsymbol{\Phi}_{\xi V} & 1 \end{bmatrix},$$

where $\boldsymbol{\Psi}_Q$ and $\boldsymbol{\Psi}_V$ were the individual covariance matrices for the random coefficients of the quantitative and verbal learning models, respectively. The submatrix $\boldsymbol{\Psi}_{VQ}$ represented the covariances between the random coefficients of the latent curve models for the two learning tasks. The variance of the latent variable was set to unity: $var(\xi) = 1$. The submatrices $\boldsymbol{\Psi}_{\xi Q}$ and $\boldsymbol{\Psi}_{\xi V}$ represented the covariances between the random coefficients of the latent curve models for the procedural tasks and the latent measure of WM.

Given the model for $\mathbf{y}_i$ in Equation 12 along with assumptions in Equations 13 and 14, the mean vector and covariance matrix of $\mathbf{y}_i$ were

$$\boldsymbol{\mu}_i = \boldsymbol{\Lambda}_i \boldsymbol{\mu}_{\varphi} \tag{15a}$$

and

$$\boldsymbol{\Sigma}_i = \boldsymbol{\Lambda}_i \boldsymbol{\Phi} \boldsymbol{\Lambda}'_i + \boldsymbol{\Theta}_\epsilon. \tag{15b}$$

This implied that the marginal distribution of $\mathbf{y}_i$ was normal with mean (Equation 15a) and covariance matrix (Equation 15b)

$$\mathbf{y}_i \sim N(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i).$$

The orders of $\boldsymbol{\mu}_i$ and $\boldsymbol{\Sigma}_i$ when data were complete were each 28, the total number of repeated measures on the two procedural tasks plus the number of manifest covariates on the WM battery. The log-likelihood function was

$$ln\ L \propto C - \frac{1}{2} \sum_{i=1}^{N} (ln|\boldsymbol{\Sigma}_i| + \mathbf{q}'_i \boldsymbol{\Sigma}_i^{-1} \mathbf{q}_i),$$

where C was a constant independent of the model parameters, and $\mathbf{q}_i = \mathbf{y}_i - \boldsymbol{\Lambda}_i \boldsymbol{\varphi}$. This form of the log-likelihood function was based on the raw data, rather than sufficient statistics such as a mean vector and covariance matrix, and so required an estimation procedure such as that described by Jennrich and Schluchter (1986). A computer program allowing one to obtain simultaneous maximum likelihood estimates of the model has been written in GAUSS Version 3.6 (Aptech Systems, 2001).[3] A path diagram of the full model is presented in Figure 3. Not represented in the figure are the correlations among the quantitative and verbal learning characteristics and the latent measure of WM.

Maximum likelihood estimates of parameters are pre-

[3] Sample data and SAS PROC NLMIXED code for fitting a version of the model are available on the Web at http://dx.doi.org/10.1037/1082-989x.9.3.xxx.supp.

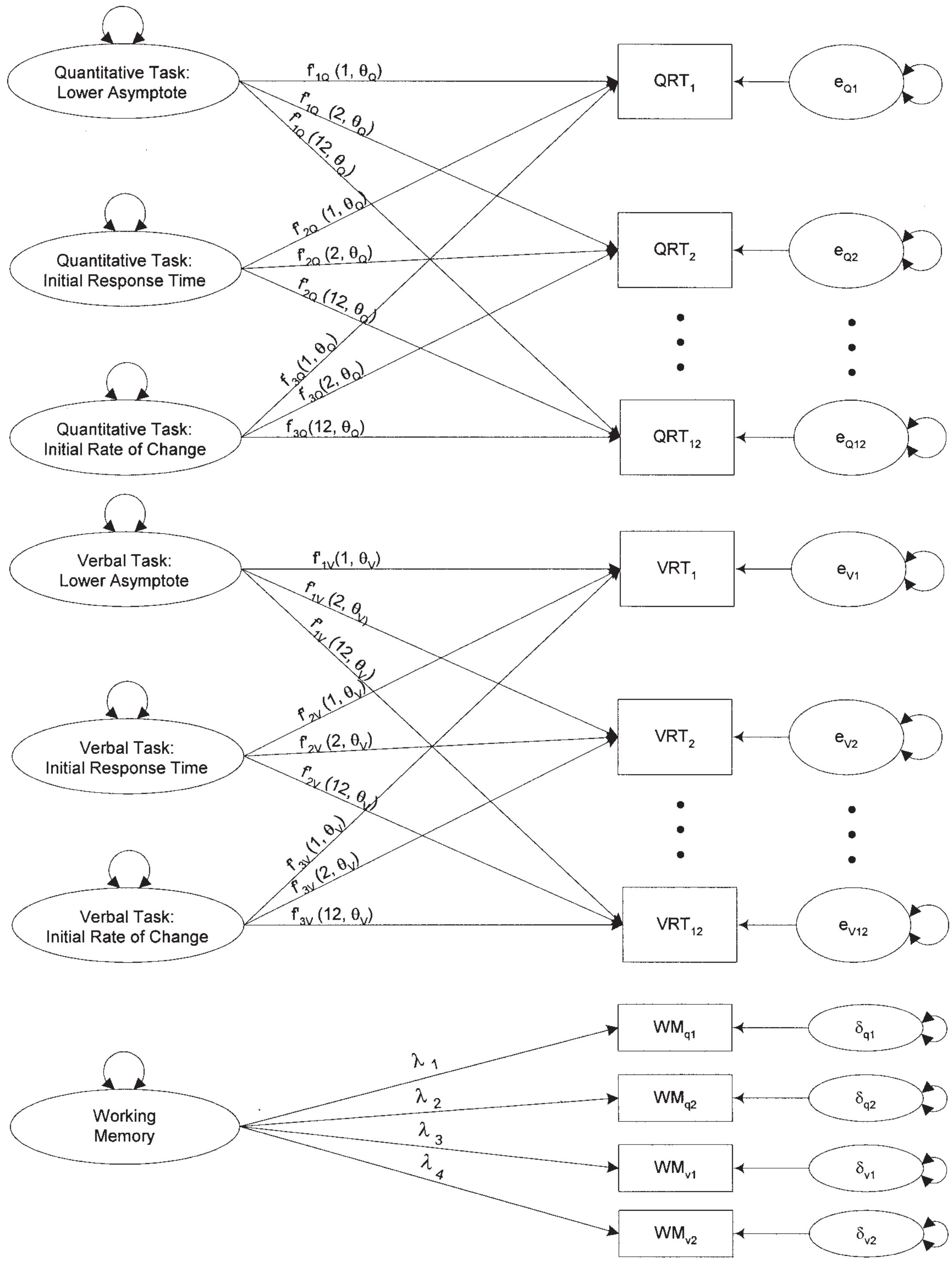


Figure 3. Path diagram of full model. QRT = quantitative response time; VRT = verbal response time; WM = working memory.

sented in the Appendix for the separate components of the full model. The estimated variances of and covariances between the random coefficients and the WM factor are presented separately in Table 1. Estimated mean initial response time scores for the quantitative and verbal tasks were 16.5 and 21.3, respectively. Estimated mean potential response time scores for the quantitative and verbal tasks were 8.63 and 6.85, respectively. Mean rates of change in

Table 1
Estimated Covariances and Correlations Among Random Coefficients and the Working Memory (WM) Factor

	Quantitative task			Verbal task			
Coefficient	Potential (η_{Q1})	Initial (η_{Q2})	Rate (η_{Q3})	Potential (η_{V1})	Initial (η_{V2})	Rate (η_{V3})	WM ξ
η_{Q1}	4.19 (0.423)	0.606	−0.148	0.633	0.420	−0.192	−0.177
η_{Q2}	7.42 (0.985)	35.8 (3.46)	0.212	0.267	0.408	−0.126	−0.016
η_{Q3}	−0.128 (0.073)	0.538 (0.205)	0.180 (0.030)	−0.133	−0.091	0.164	0.343
η_{V1}	1.66 (0.235)	2.05 (0.606)	−0.073 (0.049)	1.65 (0.207)	0.307	−0.220	−0.216
η_{V2}	8.24 (1.47)	23.4 (4.20)	0.372 (0.320)	3.78 (0.976)	92.1 (8.82)	−0.094	0.127
η_{V3}	−0.205 (0.082)	−0.392 (0.232)	0.036 (0.018)	−0.147 (0.059)	−0.468 (0.391)	0.272 (0.044)	0.145
ξ	−0.363 (0.173)	−0.093 (0.519)	0.146 (0.041)	−0.278 (0.125)	1.22 (0.781)	0.076 (0.048)	1.0

Note. Variances are on the diagonal, covariances are below the diagonal, and correlations are above the diagonal. Standard errors are in parentheses. The variance of the WM factor was set to unity.

the quantitative and verbal tasks were .701 and .721, respectively. The estimated coefficients were all large relative to their estimated standard errors. As a result, mean performance on each task was given by the estimated equations:

$$\hat{\mu}_{Qti} = 8.63 + 7.87\, exp[-.701(t-1)]$$

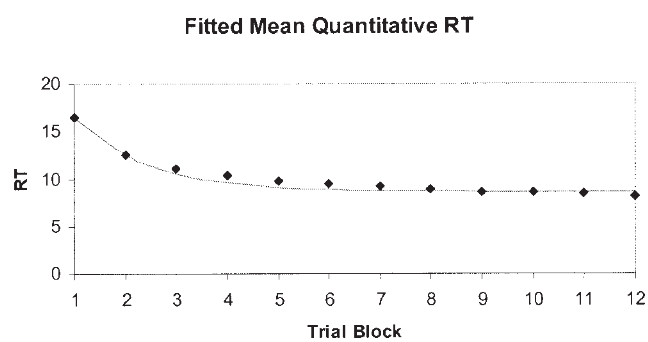
and

$$\hat{\mu}_{Vti} = 6.85 + 14.4\, exp[-.721(t-1)].$$

Plots of the fitted mean curves with the observed mean curves are presented in Figure 4.

In the case of a model for a single procedural task, the basis curves considered across the 12 trial blocks in conjunction with the individual-level random coefficients dictate the actual shape of an individual's true curve. Plots of the basis curves using estimated values for quantitative learning are given in Figure 5. The plots were comparable for both procedural tasks, so only those for the quantitative task are shown. The basis curve for the lower response time asymptote increased monotonically from zero at the first trial block, rising quickly toward an asymptote of one by about the 12th trial block. The basis curve corresponding to initial response time decreased monotonically from one at the first trial block, falling quickly toward an asymptote of zero by about the 12th trial block. The estimated mean values for the random coefficients corresponding to the first two basis curves were positive such that the contributions of the basis curves to the true scores for individuals tended to be in the same direction as those presented in Figure 5. Finally, the basis curve for initial rate of change in response times decreased quickly from zero to about a value of −4 shortly after the second trial block and then gradually increased back toward an asymptote of zero. The mean value for the random coefficients corresponding to the third basis curve was zero, suggesting that approximately half of the individuals had contributions from the third basis curve that were opposite in sign.

Individual differences in the three aspects of response time for a given task can be studied by examination of the variances of the random coefficients. The estimated variances of the random coefficients for each task were all large relative to their estimated standard errors, suggesting, for each task, individual differences in these aspects of response times. As might be done in an analysis in which a single repeated measure is considered, it is also interesting to examine the covariances between the different random coefficients within tasks. For the quantitative procedural task, latent measures of initial and potential response times had



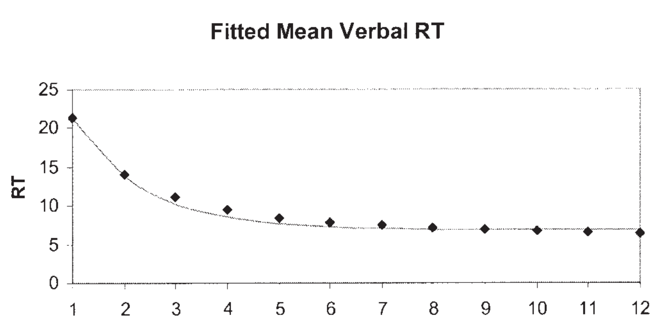


Figure 4. Fitted trial block means on the quantitative (top) and verbal (bottom) procedural tasks. RT = response time.

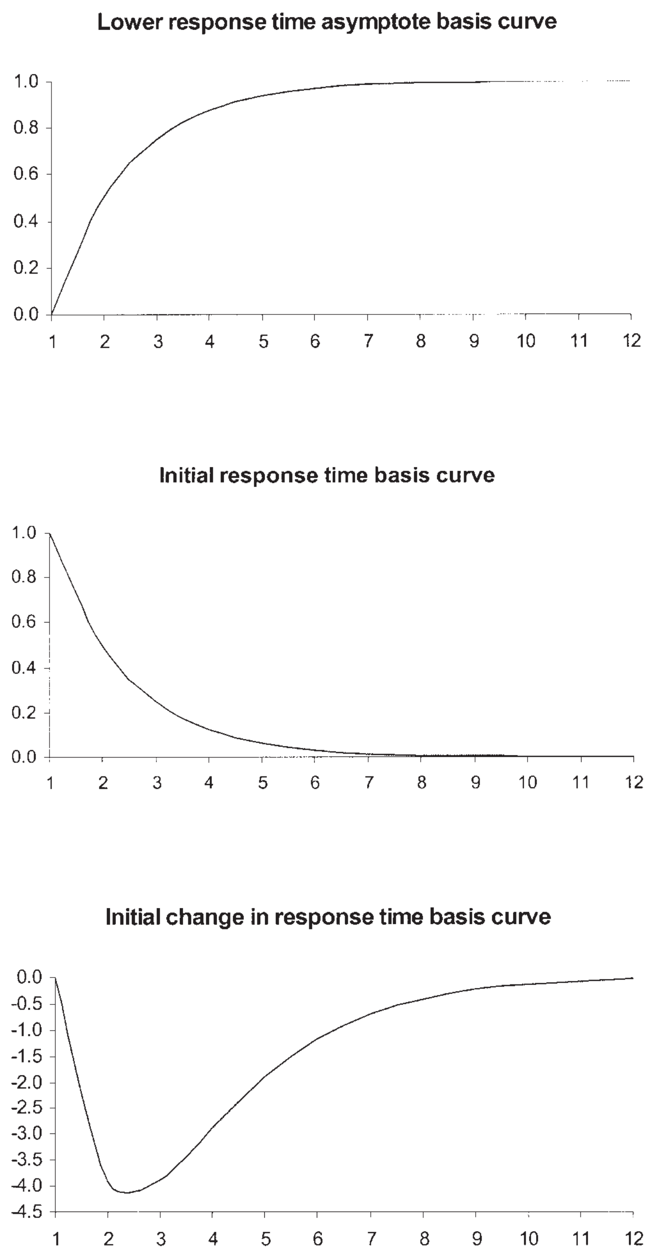


Figure 5. Basis curves for the quantitative procedural learning task.

covariance 7.42, with a corresponding correlation of .61, suggesting a moderate tendency for individuals with high initial response times to also have high potential response times. For this task, the estimated covariances between initial rate of learning and initial and potential performance levels were −.148 and .212, respectively. The correlation between initial rate of learning and initial performance was .21, suggesting a weak but positive association between these two aspects of performance: Individuals whose initial response time scores indicated fast performance at the start of the trials tended to also have an initially fast rate of change. The estimated 90% confidence interval for the covariance between learning rate and potential performance included zero as an interior point, suggesting no association between these aspects of response times. For the verbal task, initial and potential response times had covariance 3.78 with a corresponding correlation of .31, suggesting some tendency for individuals with high initial verbal response times to also have high verbal potential response times. The correlation between initial rate of change and potential performance was −.22, suggesting a weak but negative association between these two aspects of performance: Individuals who performed initially at a relatively fast rate tended to also have low potential response times.

In evaluating the fit of the models to the procedural learning data, fitted curves were compared with the observed values on both tasks.[4] Here a selection of individual curves based on performance measures of the quantitative task is shown in Figures 6 and 7, with the three best fitting and the three worst fitting curves displayed. As can be seen in Figure 6, the best fitting curves differed somewhat from each other, with one showing very little fluctuation across trial blocks and the other two, although beginning at different levels at the first trial block, showing fairly smooth decreases in response time, albeit at different rates, across trial blocks; both curves leveled off by the latter part of the trial period. In Figure 7, the top two displays represent the worst fitting curves, characterized by great fluctuations in performance during the earlier trial blocks, and then reaching relative stability in the latter half of the trials. It is interesting to note the general pattern of performance for these two cases in particular. That is, response time was relatively low at the start of the task, increased somewhat in the trials immediately succeeding that task, and then gradually dropped over the remaining trials, a pattern very different from the mean curve. Individual learning curves that show an initial slowing in performance, followed by an increase in response time, exhibit learning rates that are relatively large and positive. Finally, the third case is characterized by fairly wild fluctuations throughout the process and is not well fit by the model.

In treating the repeated measures of the two procedural learning tasks simultaneously, it is of special interest to study the estimated covariances between similar characteristics of response time on each task, in that each represents a different type of procedural learning. The estimated co-

[4] Estimates of the random coefficients were based on the joint multivariate normal distribution of $\mathbf{y}_i$ and $\boldsymbol{\eta}$ (see Davidian & Giltinan, 1995, Section 3.3; Vonesh & Chinchilli, 1997, Section 6.3). The random coefficients were the conditional expectation of $\boldsymbol{\eta}|\mathbf{y}_i$, where an estimate of $\boldsymbol{\eta}$ was the conditional expectation with all parameters set at their maximum likelihood estimates: $\hat{\boldsymbol{\eta}} = \hat{\boldsymbol{\Psi}}\hat{\boldsymbol{\Lambda}}_i'\hat{\boldsymbol{\Sigma}}_i^{-1}\mathbf{y}_i$.

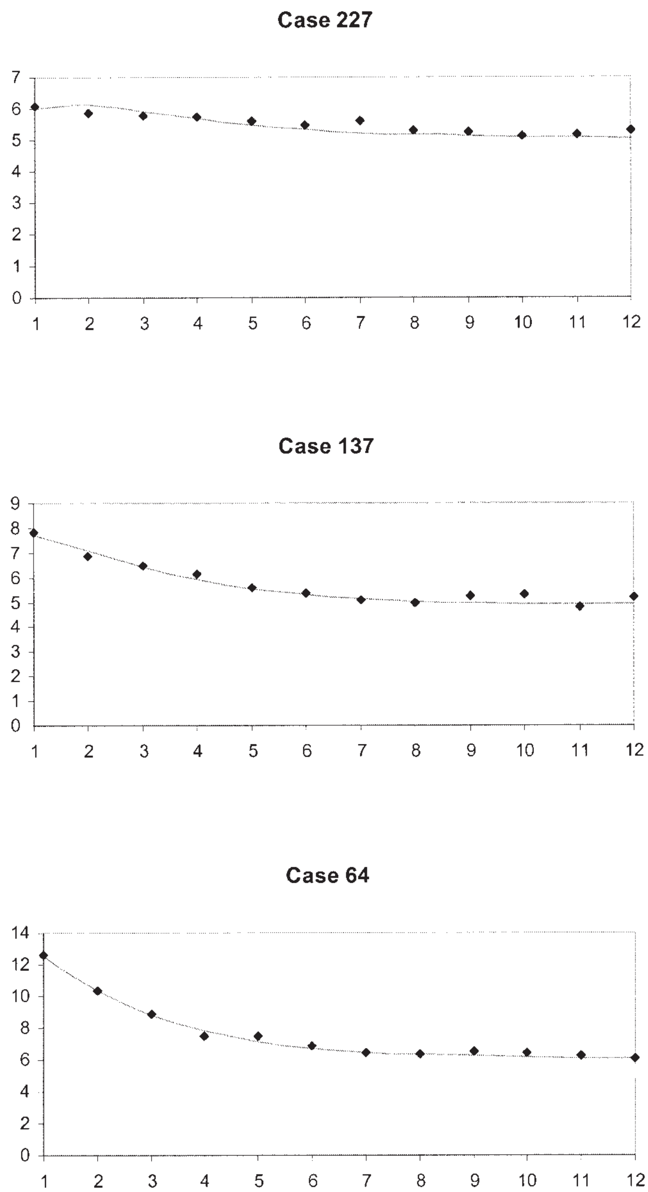


Figure 6. Three best fitting curves for the quantitative procedural learning task.

variance between initial performance on the two tasks was 23.4, with a corresponding correlation of .41. This suggests a tendency for individuals with high initial quantitative response times to also have high initial verbal response times. The moderate correlation between the two estimates of initial performance suggests that both share a quality inherent to procedural learning but that each may also capture a quality unique to the type of learning (i.e., quantitative vs. verbal). A similar pattern was found for potential learning performance. The estimated covariance between potential performance on both tasks was 1.66, with a correlation of .63, suggesting a tendency for individuals with lower asymptotic quantitative response times to also have lower asymptotic verbal response times. The moderate correlation between the two estimates of potential performance was consistent with the findings for initial performance levels. The estimated covariance between rates of learning on the two tasks was 0.036, with a correlation of .16. This suggests a tendency for individuals with fast response rates on the quantitative task to also have fast response rates on the verbal task, although the correlation itself was weak. One may also go on to investigate the associations between different aspects on different tasks (e.g., quantitative potential performance and verbal initial performance).

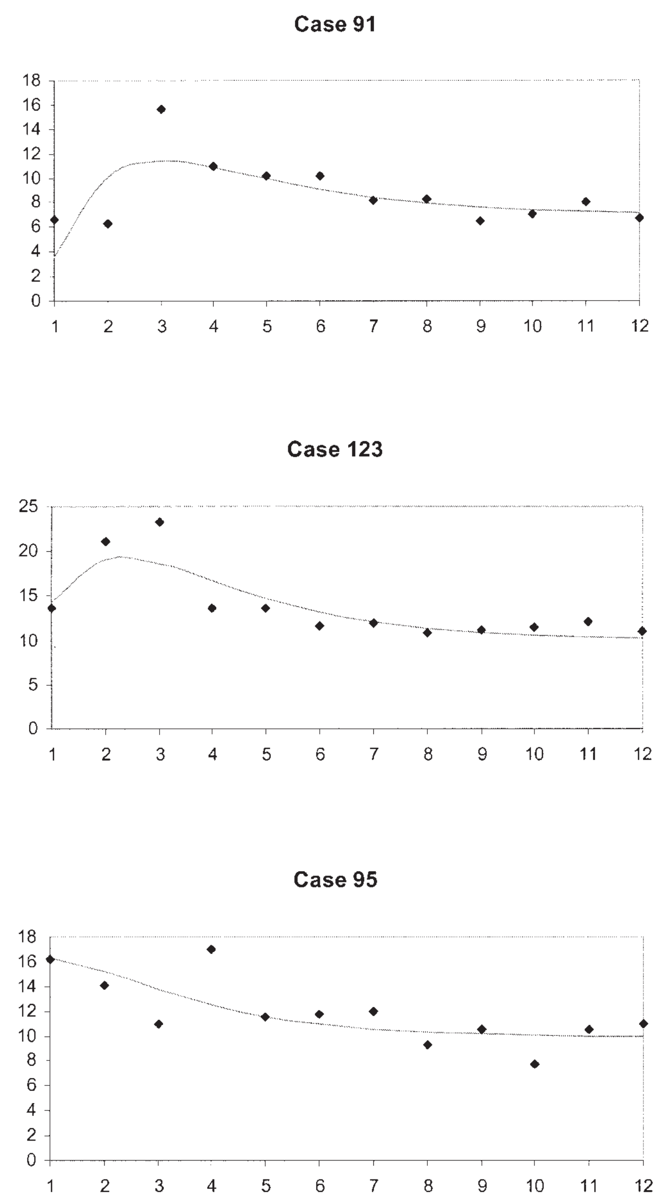


Figure 7. Three worst fitting curves for the quantitative procedural learning task.

The latent measure of WM had an estimated mean of 6.32 and a standard error of 0.44. Estimates of the associations between WM and the random coefficients of the latent curve model are examined here. The WM factor and the random coefficients relating to initial performance on the quantitative and verbal procedural tasks had covariances of −0.093 and 1.22, respectively. The estimated 90% confidence intervals for the covariances between the latent measure of WM and initial performance levels on both learning tasks included zero as interior points, suggesting no linear associations between WM and initial learning performance on either procedural task. The WM factor and the random coefficients relating to potential performance on the quantitative and verbal procedural tasks had covariances of −0.363 and −0.278, respectively, with corresponding correlations of −.18 and −.22, respectively. These correlations suggest a tendency for individuals with high scores on the latent measure of WM to also have relatively low potential response times on both procedural tasks, although these correlations were relatively weak. Finally, the WM factor and the random coefficients relating to the initial rate of change in response times on the two procedural tasks had covariances of 0.146 and 0.076, respectively. The correlation corresponding to the covariance between WM and rate of learning on the quantitative task was .34, suggesting a tendency for individuals with a high latent WM score to also exhibit high initial rates of change in quantitative response times. The estimated 90% confidence interval for the covariance between WM and initial rate of change in verbal response times included zero as an interior point, suggesting no linear association between WM and rates of change in verbal response times.

Discussion

This article has discussed the extension of the structured latent curve model for the analysis of two or more repeated measures variables that also includes a factor analysis model for covariates related to the random coefficients at the second level of the latent curve model. The multivariate form of the model is useful in studies in which multiple measures, each possibly characterized by a nonlinear form of change, are observed over time and it is of interest to study the patterns of covariation among the different change features in the set of repeated measures. In a structured latent curve model, the mean response is assumed to follow a prespecified function that may include parameters that enter in a nonlinear manner. A first-order Taylor polynomial taken about the mean function is then used to define columns of a factor matrix. The columns of the factor matrix, referred to as basis curves, define different aspects of change in the response variable over time. The parameters of the factor matrix, some of which may enter nonlinearly, are fixed across individuals. Individual latent curves are assumed to be a linear combination of the basis curves and a set of random coefficients that may be unique to each individual. This means that the basis curves may be weighted differently from one person to another, resulting in curves that may vary with regard to the different aspects of change. The result is that the individual latent curves may differ in form from the mean curve. The random coefficients of the model enter linearly, allowing estimation of the model to proceed through the use of methods typical for linear models. This kind of model is considered to be conditionally linear with regard to its random coefficients (Blozis & Cudeck, 1999).

The random coefficients also share the same substantive interpretation as the coefficients that define the mean function; that is, they define different characteristics of change but do so at the individual level. The variances of the random coefficients measure the degree of individual differences in change characteristics. Covariances among the random coefficients are measures of the linear associations among them. With the addition of a factor analysis model for covariates, it is also possible to study the associations between characteristics of change in the repeated measurements and a set of latent covariates.

The particular choice of a model when change is nonlinear is important when the focus of the analysis is on individual differences in change. It is worth keeping in mind, however, that the model one chooses for representing change in a variable is probably at best an approximation. A model is perhaps most useful when its parameters are directly interpretable in light of the behavior under investigation. A common choice for fitting nonlinear responses is a polynomial, often because polynomials are easy to estimate with standard estimation procedures. Cudeck and du Toit (2002) discussed alternative ways in which a common quadratic function may be transformed to yield different interpretations of model parameters. In cases in which a polynomial is not suitable, a nonlinear function may be preferred. Many different nonlinear response forms are possible, offering the practitioner greater flexibility over polynomial functions (see Gallant, 1987; Pinheiro & Bates, 2000). However, handling individual differences in the coefficients of a nonlinear model can be problematic. That is, random coefficients that enter the model in a nonlinear way introduce estimation difficulties not found with linear models owing to difficulties in evaluating the marginal likelihood. Recent efforts, however, have made progress in this area. Procedures such as those discussed by Davidian and Giltinan (1995) and Cudeck and du Toit (2003) approach the problem by direct maximization of the marginal likelihood. Other approaches to the estimation of nonlinear models with random coefficients have relied on linear approximations (e.g., Lindstrom & Bates, 1990). The method discussed here is a restricted form of a nonlinear random coefficient model in that the random coefficients may enter the model only in a linear manner, thus eliminating related estimation difficulties.

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Appendix

Maximum Likelihood Estimates for the Full Model

Quantitative Procedural Learning Task

Potential performance: $\hat{\theta}_{Q1} = 8.63$ (0.143).
Initial performance: $\hat{\theta}_{Q2} = 16.5$ (0.405).
Initial rate: $\hat{\theta}_{Q3} = 0.701$ (0.024).
Within-individual error variance: $\hat{\sigma}^2_Q = 1.12$ (0.056).
Within-individual error autocorrelation: $\hat{\rho}_Q = .315$ (.035).

Verbal Procedural Learning Task

Potential performance: $\hat{\theta}_{V1} = 6.85$ (0.100).
Initial performance: $\hat{\theta}_{V2} = 21.3$ (0.661).
Initial rate: $\hat{\theta}_{V3} = 0.721$ (0.020).
Within-individual error variance: $\hat{\sigma}^2_V = 1.97$ (0.092).
Within-individual error autocorrelation: $\hat{\rho}_V = .276$ (.036).

Working Memory Battery

Factor mean: $\hat{\xi} = 6.32$ (0.442).

Factor loadings: $\hat{\lambda}_1 = 12.4$ (0.875), $\hat{\lambda}_2 = 12.5$ (0.880), $\hat{\lambda}_3 = 12.5$ (0.881), $\hat{\lambda}_4 = 12.5$ (0.889).

Uniquenesses: $\hat{\psi}_1 = 223$ (27.3), $\hat{\psi}_2 = 211$ (26.3), $\hat{\psi}_3 = 208$ (26.2), $\hat{\psi}_4 = 341$ (38.7).

Note. Standard errors are in parentheses.

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